

Linearized general relativity and the Lanczos potential

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Abstract

Recently, there has been a revival of interest in the Lanczos potential of the Weyl conformal tensor. Previous work by Novello and Neto has been done with the linearized Lanczos potential as a model of a spin-2 field, which depends on a massless limit of the field. In this paper, we look at an action based on a massless potential, and show that it is classically equivalent to the linearized regime of general relativity, without reference to any limiting case.

1 Introduction

It is well-known that the quantization of gravity has been a challenging problem unsolved for several decades. There have been a number of attempts to rewrite general relativity in terms of different variables to provide an alternate viewpoint that might prove fruitful. In general, these new ideas – such as connection variables, twistors, and the null surface formalism – are based on geometric notions. However, we could also follow the lead of Feynman [4]: start with a linear action for a spin-2 field, and build up general relativity by requiring consistency and gauge invariance. Certainly, in any tensor with more than two indices, we can find a spin-2 piece to work with. Yet we can combine these two approaches, since there exists a spin-2 piece of a tensor (in addition to the metric) that has a geometric interpretation – namely, the Lanczos potential L_{abc} [7]. On a general spacetime, it has been shown [2, 6] that the Weyl tensor can be written as¹

$$C_{abcd} = 2L_{ab[c;d]} + 2L_{cd[a;b]} - g_{a[c}(L_{|b|d]} + L_{d|b]) + g_{b[c}(L_{|a|d]} + L_{d|a]) + \frac{2L}{3}g_{a[c}g_{d]b} \quad (1.1)$$

where we have defined

$$L_{ab} \equiv L_a{}^c{}_{b;c} - L_a{}^c{}_{c;b} = 2L^c{}_{a[c;b]} \quad L \equiv L^a{}_a = 2L^{ab}{}_{[a;b]}$$

Equation (1.1) is analogous to the construction of the Maxwell field from the vector potential. The Lanczos potential has the symmetries

$$L_{abc} = -L_{bac} \quad L_{[abc]} = 0$$

There are two additional properties that are usually assumed – the *algebraic gauge* $L^{ab}{}_b = 0$ and the *differential gauge* $L_{abc}{}^{;c} = 0$. The proof of the relation (1.1) given by Bampi and Caviglia does not depend on the choice of either gauge²; the spinorial proof of Illge assumes the algebraic gauge, since in this case, the

¹We use the notation

$$A_{(ab)} = \frac{1}{2}(A_{ab} + A_{ba}) \quad A_{[ab]} = \frac{1}{2}(A_{ab} - A_{ba})$$

with a comma denoting partial, and a semicolon covariant, differentiation.

²In most papers on the Lanczos potential, the word "gauge" is used rather loosely to reflect the freedom to pick the values of the tensors $L^{ab}{}_b$ and $L_{abc}{}^{;c}$. However, there is a precise geometric meaning behind this, as shown by Hammon and Norris [5], namely, that we can think of L_{abc} as a certain piece of a connection on an affine frame bundle over spacetime with structure group $GL(4) \ltimes T_2^1\mathbb{R}^4$. The fact that the Lanczos potential is related to the translations $T_2^1\mathbb{R}^4$ (and hence is Abelian) gives rise to the fact that the Weyl tensor is linear in L_{abc} .

Lanczos potential is associated with a spinor $L_{ABCC'} = L_{(ABC)C'}$, a natural choice for a spin-2 field. For the rest of this paper, we will assume that both gauge conditions hold, so that $L_{ab} = L_a{}^c{}_{b;c}$ and $L = 0$.

Now we can consider linear actions of the Lanczos potential as an alternate model for gravity, which has been done previously by Novello and Neto [8, 9]. However, their work starts with an action for a massive field, followed by a massless limit to recoup linearized general relativity. This limit becomes problematic when one has to consider the interaction of L_{abc} with matter; as explained below, the massless limit seems to imply that interactions in the Lanczos potential picture correspond to no interaction in the related metric picture. The main goal of this paper is to relate the two models of linearized general relativity using perturbations of the Lanczos potential and the metric without relying on a massless limit. By using some recent results of Andersson and Edgar [1], we can derive the relation without any extra conditions on the perturbations.

2 Linear spin-2 fields

On a flat background, we can consider the spacetime metric to be of the form

$$g_{ab} = \eta_{ab} + h_{ab}$$

where η_{ab} is the flat metric, and write out the form of the conformal tensor C_{abcd} . When we compare this form to that in terms of the Lanczos potential (1.1), choosing the gauge

$$h^{ab}{}_{,b} = \frac{1}{2}h^{,a} \quad (2.1)$$

with $h = h^a{}_a$, and using the tensor η^{ab} to raise and lower indices, we have that [7]

$$L_{abc} = \frac{1}{2} \left(h_{c[a,b]} - \frac{1}{6} \eta_{c[a} h_{b]} \right) \quad (2.2)$$

This expression³, valid to linear order, is a special case of a more general theorem [1]. Suppose we have a tensor with the symmetries of the Lanczos potential; then, on a Ricci-flat spacetime with Lorentz signature metric g_{ab} , we can locally find a traceless tensor \tilde{K}_{ab} , such that

$$L_{abc} = \tilde{K}_{c[a;b]} + \frac{1}{3} g_{c[a} \tilde{K}_{b]d}{}^{;d} \quad (2.3)$$

(assuming \tilde{K}_{ab} is real). Since \tilde{K}_{ab} is traceless, we can write it as $\tilde{K}_{ab} = K_{ab} - \frac{1}{4} g_{ab} K$, where $K = g^{ab} K_{ab}$. Then, (2.3) becomes

$$L_{abc} = K_{c[a;b]} - \frac{1}{3} g_{c[a} K_{b]} + \frac{1}{3} g_{c[a} K_{b]d}{}^{;d} \quad (2.4)$$

With $K_{ab} = \frac{1}{2} h_{ab}$ and using (2.1), we arrive at (2.2).

Now we briefly outline the work of Novello and Neto. By writing down an action made of massive fields h_{ab} and L_{abc} [9] (again using the flat metric to raise indices),

$$S_1 = \int \left[h^{ab} (L_a{}^c{}_{b;c} - Q L_a{}^c{}_{c,b} - Q \eta_{ab} L^{ab}{}_{a,b}) + \frac{1}{2} m^2 h^{ab} h_{ab} + \frac{1}{4} L^{abc} L_{abc} - \frac{B-Q}{2(1-3B)} L^{ac}{}_{c,b} L_a{}^b{}_{,b} \right] \quad (2.5)$$

where $Q = \frac{1-B}{1-3B}$ and B is arbitrary, we find that the equations of motion are

$$L_{abc} = 2(h_{c[a,b]} - B \eta_{c[a} h_{b]} + B \eta_{c[a} h_{b]d}{}^{;d}) \quad (2.6)$$

³As an aside, we note that this relation illuminates the discussion in Section 5.7 of Penrose and Rindler [10], and especially the origin of equations such as (5.7.12) in that work.

and

$$h_{ab} = -\frac{1}{m^2}L_{(a}{}^c{}_{b),c} + \frac{Q}{m^2}L_{(a}{}^c{}_{|c|,b)} + \frac{1}{3m^2}\eta_{ab}L^{cd}{}_{c,d} \quad (2.7)$$

Note that (2.6) has the same form as (2.4) when $B = 1/3$. Because we now have expressions for each of the fields in terms of derivatives of the other, we can select either one to be an auxiliary field, and substitute the appropriate formula into the action, giving a Lagrangian for a single field. Solving for L_{abc} in terms of h_{ab} gives the usual linearized Einstein-Hilbert action with a mass term, while the reverse results in

$$S_2 = \int \left[\frac{1}{16}C^{abcd}C_{abcd} + \frac{1}{2}m^2L^{abc}L_{abc} \right] \quad (2.8)$$

where C^{abcd} is given in terms of the linearized Lanczos potential L_{abc} as in (1.1). Varying with respect to the potential gives

$$C^{abcd}{}_{,d} + m^2L^{abc} = 0 \quad (2.9)$$

We cannot take the massless limit here if we started with (2.5), since then, we would be unable to get the relation (2.7), and hence we could not solve for h_{ab} in terms of L_{abc} to arrive at (2.8). However, if we take (2.8) as our starting point, then there is no difficulty in taking a massless limit. The limit does cause problems when one tries to add the interaction of L_{abc} to matter. We can certainly add a term of the form

$$S_{int} = k' \int L_{abc}J^{abc} \quad (2.10)$$

but then the question arises of how to relate the matter current J_{abc} to the usual energy-momentum tensor T_{ab} . With a massive L_{abc} , we can use (2.7) and start with the interaction in the metric picture to get

$$S_{int} = k \int h_{ab}T^{ab} = \frac{k}{m^2} \int \left[-L_{(a}{}^c{}_{b),c} + QL_{(a}{}^c{}_{|c|,b)} + \frac{1}{3}\eta_{ab}L^{cd}{}_{c,d} \right] T^{ab} \quad (2.11)$$

As we can see, there are hazards in taking the massless limit – from (2.11), it seems that taking the limit such that k/m^2 remains finite is equivalent to the limit $k \rightarrow 0$, i.e. in the metric picture, there is no interaction between matter and the graviton⁴. To avoid this, we start from a different viewpoint. We can certainly write down the action (2.8) for a spin-2 field and find its equations of motion with $m = 0$. The question is how we can relate the solutions L_{abc} with the solutions h_{ab} of the linearized Einstein equations. Because of the theorem of Andersson and Edgar, we will always be able locally to find a symmetric tensor h_{ab} such that its derivatives, when combined in the form (2.4), give a massless solution L_{abc} . From these equations of motion (2.9), it is natural to say that L_{abc} is a perturbation of the Lanczos potential, implying h_{ab} is a perturbation of the metric. Then we have the following result.

Proposition 2.1 *Every solution L_{abc} of (2.9) with $m = 0$ corresponds to a solution h_{ab} of the linearized Einstein-Hilbert action, up to gauge transformations of h_{ab} .*

Proof. To show that h_{ab} gives rise to a solution L_{abc} , we use (2.2) to form the linearized Lanczos potential in the gauge (2.1). To go in the opposite direction, suppose that h_{ab} and h'_{ab} are two metric perturbations that give the same Lanczos potential L_{abc} . Then, $\Delta h_{ab} \equiv h'_{ab} - h_{ab}$ gives a Lanczos potential of zero (implying the Weyl tensor is zero) and satisfies the field equations $R_{ab}(\Delta h_{cd}) = 0$. Because of this, the Riemann tensor constructed from Δh_{ab} is identically zero; hence, Δh_{ab} corresponds to a constant multiple of a flat metric to linear order, which can be absorbed by a gauge transformation. ■

To see the relation between the two models with the addition of matter, we can introduce a coupling to a matter current of the form (2.10). Using ideas similar to those above, we find the relation between J_{abc} and

⁴In another paper [8], the starting point of Novello and Neto is (2.8), but varying both the metric and the Lanczos potential. However, since an equation like (2.7) has to be used to get rid of the dependence on the metric, then the same problem results.

the more usual matter tensor T_{ab} without reference to the mass. Using (2.4) to write the current in terms of a symmetric tensor T_{ab} , we will show that T_{ab} acts like as a energy-momentum tensor of matter. Assuming that L_{abc} is in the algebraic gauge, then (2.10) is invariant under the transformation $J^{abc} \rightarrow J^{abc} + \eta^{c[a} V^{b]}$. Thus, by choosing $V^a = -\frac{1}{3} T^{ad}_{,d}$, we have that

$$J^{abc} = T^{c[a, b]} - \frac{1}{3} \eta^{c[a} T^{, b]}$$

and the equations of motion become

$$C^{abcd}_{,d} = J^{abc} = T^{c[a, b]} - \frac{1}{3} \eta^{c[a} T^{, b]} \quad (2.12)$$

These equations, along with the Einstein equations and the Bianchi identities, written solely in terms of the metric for the Weyl tensor,

$$C^{abcd}_{,d} = R^{c[a, b]} - \frac{1}{6} \eta^{c[a} R^{, b]}$$

lead to the identification of the tensor T^{ab} as the energy-momentum tensor. The fact that the divergence of T^{ab} is zero comes from the trace-free nature of the left-hand side of (2.12):

$$\begin{aligned} C^{ab\ d}_{,d} &= 0 = J^a_b \\ &= \frac{1}{2} \left[T^{ab}_{,b} - T^{,a} - \frac{1}{3} T^{,a} + \frac{4}{3} T^{,a} \right] \\ &= \frac{1}{2} T^{ab}_{,b} \end{aligned}$$

If we start with the same action (2.8) on a curved background, we find that the equations of motion are inconsistent unless the Weyl tensor of the background spacetime is zero. However, this is not a disability, but rather a sign that, if we consider the gravitational degrees of freedom to reside in the conformal curvature, then it does not make sense to start with a background that already has gravitational interactions.

3 Discussion

In this paper, we have shown that there is a relationship between linearized general relativity written in terms of perturbations of the metric and the Lanczos potential, when expanded on a flat background spacetime. Unlike in [8, 9], there is no need to solve for bridging formulas between the two fields using the equations of motion, so this relationship holds regardless of the mass of the field. Because our action is written only with the linearized Lanczos potential, one can hope that the full non-linear theory can be more easily arrived at. As a sketch of the procedure, we note that there are four curvature invariants that can be formed out of the Weyl tensor, namely,

$$C_{abcd} C^{abcd}, \quad {}^*C_{abcd} C^{abcd}, \quad C_{abcd} C^{cd}_{ef} C^{abef}, \quad {}^*C_{ab}{}^{cd} C_{cd}{}^{ef} C_{ef}{}^{ab}$$

(The quantity ${}^*C_{abcd}$ is the dual of the Weyl tensor, ${}^*C_{abcd} = \frac{1}{2} \epsilon_{abef} C^{ef}_{cd}$). Thus, any analytic Lagrangian must be written as a polynomial in these four quantities. Because we can extend (2.2) to arbitrary order in h_{ab} , by substitution, one can relate the equations of motion of such an action to the Einstein equations. Another, perhaps more elegant, method is to use the ideas of Deser [3], deriving an action to all orders from the requirements of consistency and gauge invariance. We hope to return to this problem in the future.

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